Toy Model 1D

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Abstract

In this note, we try to understand what are the factors involved in the statistics of the power spectra of noise simulations with a non-uniform weighting. We use a simple 1D model to simplify the maths.

1 Simple simulations

We generate a vector x_i of $\mathrm{N_{modes}} = 10^4$ random variables $\mathcal{N}(\mu_n, \sigma_n)$. We will use $\mu_n = 0$ and $\sigma_n = 1$.

We then create different types of hitmaps h_i (shown in the posting).

We compute noise vectors:

$$n_i = \frac{x_i}{\sqrt{h_i}} \tag{1}$$

These represent noisy data points of variance $1/h_i$.

We also generate a signal vector s_i from Gaussian random variables $\mathcal{N}(0, \sigma_s)$.

2 Power estimation

We design an estimator of the bandpower BP of a vector v :

$$\mathrm{BP} =< v_i^2 >_{w_i^2}= \frac{\sum_i w_i^2 v_i^2}{\sum_i w_i^2}, \tag{2}$$

The weights w_i are to be chosen, we will later use the hitcount map as it represents the inverse variance of the noise.

Our goal in this note is to derive the variance of the bandpowers:

$$\mathrm{Var(BP)} =<< \mathrm{BP}^2 > - < \mathrm{BP} >^2> . \tag{3}$$

Let's first compute $< \mathrm{BP}^2 >$:

$$< \mathrm{BP}^2 >=< \left(\frac{\sum_i w_i^2 v_i^2}{\sum_i w_i^2}\right)^2 > =< \frac{\left(\sum_i w_i^4 v_i^2\right)^2}{\left(\sum_i w_i^2\right)^2} >=< \frac{\sum_i w_i^4 v_i^4 + \sum_{i\neq j} w_i^2 v_i^2 w_j^2 v_j^2}{\left(\sum_i w_i^2\right)^2} > \tag{4}$$

and $< \mathrm{BP} >$

$$< \mathrm{BP} >=< \frac{\sum_i w_i^2 v_i^2}{\sum_i w_i^2} > \tag{5}$$

Then

$$\mathrm{Var(BP)} =< \frac{\sum_i w_i^4 h_i^{-2} x_i^4 + \sum_{i\neq j} w_i^2 h_i^{-1} x_i^2 w_j^2 h_j^{-1} x_j^2}{\left(\sum_i w_i^2\right)^2} > - \left(< \frac{\sum_i w_i^2 h_i^{-1} x_i^2}{\sum_i w_i^2} >\right)^2 . \tag{6}$$

Using the fact that since x_i and x_j are independent, $< x_i x_j >=< x_i >< x_j >$, and using the notation:

$$\mu_n =< x^n >= \frac{\sum_i x^n}{\mathrm{N_{modes}}} \tag{7}$$

then

$$\mathrm{Var(BP)} = \mu_4 \frac{\sum_i w_i^4 h_i^{-2}}{\left(\sum_i w_i^2\right)^2} - \mu_2^2 \frac{\sum_{i\neq j} w_i^2 h_i^{-1} w_j^2 h_j^{-1}}{\left(\sum_i w_i^2\right)^2} - \left(\mu_2 \frac{\sum_i w_i^2 h_i^{-1}}{\sum_i w_i^2}\right)^2. \tag{8}$$

Which can be simplified to

$$\mathrm{Var(BP)} = \mu_4 \frac{\sum_i w_i^4 h_i^{-2}}{\left(\sum_i w_i^2\right)^2} - \mu_2^2 \frac{\sum_i w_i^4 h_i^{-2}}{\left(\sum_i w_i^2\right)^2}. \tag{9}$$

2.1 Gaussian noise case

In the case where the noise is Gaussian, $\mu_4 = 3\sigma_n^4$ and $\mu_2 = \sigma_n^2$, and we get :

$$< \mathrm{BP} >=< \frac{\sum_i w_i^2 h_i^{-1} x_i^2}{\sum_i w_i^2} >= \mu_2 \frac{\sum_i w_i^2 h_i^{-1}}{\sum_i w_i^2} = \sigma_n^2 \frac{\sum_i w_i^2 h_i^{-1}}{\sum_i w_i^2} \tag{10}$$

and

$$\mathrm{Var(BP)} = 2\sigma_n^4 \frac{\sum_i w_i^4 h_i^{-2}}{\left(\sum_i w_i^2\right)^2}. \tag{11}$$

If we weigh our estimator with the hitcount map, i.e. $w_i = h_i$ (which should be the optimal choice), we get:

$$< \mathrm{BP} >= \sigma_n^2 \frac{\sum_i h_i}{\sum_i h_i^2} \tag{12}$$

and

$$\mathrm{Var(BP)} = 2\sigma_n^4 \frac{1}{\sum_i h_i^2}. \tag{13}$$

2.2 Gaussian signal case

In the case of Gaussian we have $\mu_4 = 3\sigma_s^4$ and $\mu_2 = \sigma_s^2$ and there is no hitcount weighting (see equation 1). Then we get

$$< \mathrm{BP} >=< \frac{\sum_i w_i^2 x_i^2}{\sum_i w_i^2} >= \mu_2 \frac{\sum_i w_i^2}{\sum_i w_i^2} = \sigma_s^2 \tag{14}$$

and

$$\mathrm{Var(BP)} = 2\sigma_s^4 \frac{\sum_i w_i^4}{\left(\sum_i w_i^2\right)^2}. \tag{15}$$

If we weigh our estimator with the hitcount map, i.e. $w_i = h_i$ the expression is of course simply

$$\mathrm{Var(BP)} = 2\sigma_s^4 \frac{\sum_i h_i^4}{\left(\sum_i h_i^2\right)^2}. \tag{16}$$

3 Effective f_{sky}

The effective number of degrees of freedom is given by :

$$k = 2\,\frac{< \mathrm{BP} >^2}{\mathrm{Var(BP)}} \tag{17}$$

The effective f_{sky} is then the ratio of these degrees of freedom to the total number of modes on the full sky, i.e. $\mathrm{N_{modes}}$:

$$f_{\mathrm{sky,eff}} = \frac{k}{\mathrm{N_{modes}}}. \tag{18}$$

using equation 10 and 11 (or 14 and 15), we get

$$\text{Noise} : f_{\mathrm{sky,eff}} = \frac{< w_i^2 h_i^{-1} >^2}{< w_i^4 h_i^{-2} >} \tag{19}$$

$$\text{Signal} : f_{\mathrm{sky,eff}} = \frac{< w_i^2 >^2}{< w_i^4 >} \tag{20}$$

In the case where we weigh using the hitcount map, we get:

$$\text{Noise} : f_{\mathrm{sky,eff}} = \frac{< h_i >^2}{< h_i^2 >} \tag{21}$$

$$\text{Signal} : f_{\mathrm{sky,eff}} = \frac{< h_i^2 >^2}{< h_i^4 >} \tag{22}$$